Construction of covers in positive characteristic via degeneration

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Abstract

In this note we construct examples of covers of the projective line in positive characteristic such that every specialization is inseparable. The result illustrates that it is not possible to construct all covers of the generic r-pointed curve of genus zero inductively from covers with a smaller number of branch points.

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Let k be an algebraically closed field of characteristic $p > 0$. Let $X = \mathbb{P}^1_k$ and G a finite group. We fix $r \geq 3$ distinct points $\mathbf{x} = (x_1, x_2, \ldots, x_r)$ on X. We ask whether there exists a tame Galois cover $f : Y \to X$ with Galois group G which is branched at the x_i. If p does not divide the order of G, then the answer is well known. Namely, such a cover exists if and only if G may be generated by $r-1$ elements of order prime to p.

Suppose that p divides the order of G. Then the existence of a G-cover as above, depends on the position of the branch points x_i. (See, for example, [6, Lemma 6].) In this note we restrict to the case that $(X; \mathbf{x})$ is the generic r-pointed curve of genus zero. A more precise version of the existence question in positive characteristic is whether there exists a G-Galois cover of $(X; \mathbf{x})$ with given ramification type (see for example [6]). For the particular kinds of groups we consider here, we define the ramification type in §1.

Osserman ([4]) proves (non)existence of covers in positive characteristic, for certain ramification types. His method is roughly as follows. First, he proved results for covers branched at $r = 3$ points. In this case his results are strongest. Using the case $r = 3$, he then constructs *admissible covers* of degenerate curves which deform to covers of smooth curves (see §2 for a definition).

Suppose we are given a tame G-Galois cover π of $(X = \mathbb{P}^1_k; \mathbf{x})$. Osserman asks ([4, §6]) whether there exists a degeneration $(\bar{X}, \bar{\mathbf{x}})$ of $(X; \mathbf{x})$ such that π specializes to an admissible cover of $(\bar{X}, \bar{\mathbf{x}})$. If such a degeneration exists, he says that π has *a good degeneration*. Covers which admit a good degeneration are exactly those which may be shown to exist inductively from the existence of covers with less branch points. To goal of this note is to produce covers which do not have a good degeneration. We show that such covers exist with arbitrary large number of branch points.

1 Meta-abelian covers

In this section, we recall a result from [1] on the existence of tame Galois covers with Galois group $G \simeq (\mathbb{Z}/p)^n \rtimes \mathbb{Z}/m$. Let $p \neq 2$ be a prime and m be an integer prime to p. Let f be the order of $p \pmod m$. We suppose that k is an algebraically closed field of characteristic p.

Let $\mathbf{x} = (x_1, \dots x_r)$ be r distinct k-rational points of $X = \mathbb{P}^1_k$. Let $\mathbf{a} = (a_1, \dots, a_r)$ be an r-tuple of integers with $0 < a_i < m$ and $\sum a_i \equiv 0 \bmod m$. Suppose moreover that $\gcd(m, a_1, \dots, a_r) = 1$. Let $g : Z \to X$ be the m-cyclic cover of *type* $(\mathbf{x}; \mathbf{a})$ ([1]), i.e. Z is the complete nonsingular curve given by the equation

$$z^m = \prod_{i : x_i \neq \infty} (x - x_i)^{a_i}$$

and $g : (x, z) \mapsto x$. We denote by $\sigma(Z)$ the p-rank of Z. Then $\sigma(Z) = \dim_{\mathbb{F}_p} V$, where $V := \mathrm{Hom}(\pi_1(Z), \mathbb{Z}/p)$. Since $\mathbb{Z}/m\mathbb{Z}$ acts on V, there exists a tame $G := V \rtimes \mathbb{Z}/m\mathbb{Z}$-Galois cover $\pi : Y \to X$ which factors through Z.

The following proposition gives an upper bound on $\sigma(Z)$ which is attained if the branch points x_i are sufficiently general. For a more precise version, we refer to [1]. See also [2] for the case $f = 1$. For every integer a, we denote by $\langle a \rangle$ the unique integer with $\langle a \rangle \equiv a \pmod m$ and $0 < \langle a \rangle < m$.

Let $\chi : \mathbb{Z}/m\mathbb{Z} \to \mathbb{F}_{p^f}^\times$ be a nontrivial, irreducible character. Let $I = \{1, \dots, m-1\}/\sim$, where $i \sim p^j i$. Then I corresponds to the set of nontrivial, irreducible characters $\mathbb{Z}/m\mathbb{Z} \to \mathbb{F}_p^\times$. For every $i \in I$, we let $n_i := (p^f - 1)/\gcd(i, m)$ be the number of elements of the equivalence class of i.

Proposition 1.1 *(a) We have that*

$$\sigma(Z) \leq B(\mathbf{a}) := \sum_{i \in I} n_i \min_{0 \leq i \leq f-1} (r - 1 - \frac{1}{m} \sum_{j=1}^{r} \langle p^i a_j \rangle).$$

(b) Suppose that $p \geq m(r-3)$. There exists $x_1, \dots, x_r \in X = \mathbb{P}^1_k$ such that

$$\sigma(Z) = B(\mathbf{a}).$$

Proof: Part (a) is proved in [1, Lemma 4.3]. Part (b) follows from [1, Theorem 6.1]. □

In [1] one finds some variants of this result: under certain additional hypotheses on the type, we may weaken the condition on p.

The number $r - 1 - (\sum_{j=1}^r \langle i a_j \rangle)/m$ is the dimension of the χ^{-i}th-eigenspace of $H^1(C, \mathcal{O}_C)$ ([1]). It is well-known that this number is an upperbound for the dimension of the χ^ith eigenspace of $V \otimes_{\mathbb{F}_p} \mathbb{F}_{p^f}$. The following statement immediately follows from Proposition 1.1.

Corollary 1.2 *Let $g : Z \to X$ be an m-cyclic cover of type $(\mathbf{x}; \mathbf{a})$, where $(X; \mathbf{x})$ is generic. Suppose that $p \geq m(r-3)$. Define*

$$\gamma(s) = \frac{1}{m} \sum_{t=1}^{r} \langle s a_t \rangle.$$

Then Z is ordinary if and only $\gamma(s) = \gamma(p^i s)$, for all i.

2 Degeneration

Let $R = k[[t]]$ be a discrete valuation ring of equal characteristic p and let $\mathcal{X} \to \mathrm{Spec}(R)$ be a semistable curve over R whose generic fiber is smooth. Let $x_1, \ldots, x_r : \mathrm{Spec}(R) \to \mathcal{X}$ be disjoint section, which avoid the singularities of $X_0 := \mathcal{X} \times_R k$.

Definition 2.1 Let $\pi_K : Y_K \to X_K$ be a tame cover of smooth projective curves. We say that π_K has a *good degeneration* if there exists a discrete valuation ring R with fraction field K and a finite morphism $\pi : \mathcal{Y} \to \mathcal{X}$ of semistable curves over $\mathrm{Spec}(R)$ with generic fiber π_K such that the branch locus is étale over $\mathrm{Spec}(R)$ and the special fiber is separable. If this holds, we call $\pi_R : \mathcal{Y} \to \mathcal{X}$ (or also its special fiber $\pi_0 := \pi_R \otimes_R k : Y_0 \to X_0$) a *good degeneration* of π_K.

Let $\pi_K : Y_K \to X_K$ be a tame cover of smooth projective curves which has a good degeneration. Let $\pi_R : \mathcal{Y} \to \mathcal{X}$ be as in the statement of Definition 2.1. Then the special fiber $\pi_0 := \pi \otimes_R k : Y_0 \to X_0$ is an *admissible cover*. We recall the definition and refer to [6, §2.1] for a short introduction to admissible covers. Let τ be any singularity of Y_0, and let Y_1, Y_2 be the (not necessarily different) irreducible components of Y_0 which intersect in τ. Then we require that the canonical generators h_i (with respect to some chosen system of roots of unity) of the stabilizer of $\tau \in Y_i$ satisfy $h_1 \cdot h_2 = 1$. (Recall that h is a *canonical generator* if there exists a local parameter u of τ such that $h^* u = \zeta_n \cdot u$, where n is the order of the stabilizer of τ.)

Let $G = (\mathbb{Z}/p\mathbb{Z})^n \rtimes \mathbb{Z}/m\mathbb{Z}$ and $R = k[[t]]$ and $K = k((t))$. Suppose that $\pi_K : Y_K \to X_K$ is a tame G-Galois cover which has a good degeneration. Let $\pi : \mathcal{Y} \to \mathcal{X}$ be a finite morphism as in Definition 2.1. It is easy to see if π_K has a good degeneration, then there exists a good degeneration $\pi : \mathcal{Y} \to \mathcal{X}$ such that the special fiber X_0 of $\mathcal{X}$ consists of two projective curves meeting in one point τ. We denote these components by X_1 and X_2. Write $S_i \subset \{1, \ldots, r\}$ for the indices j such that x_j specializes to X_i. We write $\pi_i : Y_i \to X_i$ $(i = 1, 2)$ for the restriction of $\pi_0 := \pi \otimes_R k$ to X_i.

Let $Z_K = Y_K/(\mathbb{Z}/p\mathbb{Z})^n$ and write $g_K : Z_K \to X_K$ for the m-cyclic cover associated to π_K. Let $(\mathbf{x}; \mathbf{a})$ be the type of g_K.

Lemma 2.2 *Let $\pi_K : Y_K \to X_K$ be a G-Galois cover as above. Suppose that $X_0 := \mathcal{X} \otimes_R k$ consists of two irreducible components X_1, X_2, as above. Then $\pi_i : Y_i \to X_i$ $(i = 1, 2)$ has type*

$$((x_j)_{j \in S_i} \cup (\tau) \,;\, (a_j)_{j \in S_j} \cup (\sum_{j \notin S_j} a_j)).$$

Proof: This follows immediately from the definition of the type. □

A well-known result of formal patching ([3], [5]) states that every tame admissible cover may be deformed to a cover of smooth curves. This may be used to produce examples of covers which have a good degeneration. For example, one easily checks the following. Let $\pi : Y \to X$ be a G-Galois cover of type $(\mathbf{x}; \mathbf{a})$, where $(X; \mathbf{x})$ is the generic r-pointed curve of genus zero. If there exists $1 \leq i < j \leq r$ such that $a_i + a_j = m$, then π has a good degeneration. We give an easy example of a cover which does not have a good degeneration.

Example 2.3 Let $m = 5$ and let $p \equiv -1 \pmod{m}$. Then the order, f, of p in $\mathbb{Z}/m\mathbb{Z}^*$ is 2. We consider $\mathbf{a} = (1, 1, 1, 2)$. One computes that $B(\mathbf{a}) = 2$. Proposition 1.1 implies that for p sufficiently large there exists a tame $G = (\mathbb{Z}/p\mathbb{Z})^2 \rtimes \mathbb{Z}/m\mathbb{Z}$-Galois cover $\pi : Y \to \mathbb{P}^1$ branched at 4 points which factors through a cover of type $\mathbf{a}$. In fact, [1, Proposition 7.8] implies that we do not need the lower bound on p in this case.

Let $\pi_0 : Y_0 \to X_0$ be a degeneration of π. Then X_0 consists of 2 irreducible components, which we denote by X_1 and X_2. To each of these components specialize two of the points $x_1, \ldots, x_4$. Lemma 2.2 implies that (up to renumbering) the restrictions π_1 and π_2 of π_0 would have type $\mathbf{a}_1 = (1, 1, 3)$ and $\mathbf{a}_2 = (2, 1, 2)$. One computes that $B(\mathbf{a}_i) = 0$ for $i = 1, 2$. Hence π_0 is inseparable.Therefore π does not have a good degeneration.

Remark 2.4 Suppose that $p \equiv 1 \pmod{m}$ and let $(X = \mathbb{P}^1_k; \mathbf{x})$ be the generic r-pointed curve of genus zero. Then it is shown in [1, Proposition 7.4] that every m-cyclic cover of $(X; \mathbf{x})$ has a good degeneration. Moreover, in this case we have that $B(\mathbf{a}) = g(Z)$, for every type m-cyclic cover $Z \to X$ of type $(\mathbf{x}; \mathbf{a})$.

In the case that $p \equiv -1 \pmod{m}$ it is shown in [1, Proposition 7.8] that every m-cyclic cover of $(X = \mathbb{P}^1_k; \mathbf{x})$ has a good degeneration, provided that the number of branch points is at least 5. The proof of this result relies essentially on the fact that the group scheme $J(Z)[p]$ of p-torsion points of an m-cyclic cover of $(X = \mathbb{P}^1_k; \mathbf{x})$ is self-dual under Cartier duality. The examples from §3 suggest that a similar result does not hold for $f > 2$, see Remark 3.5.

3 Covers without a good degeneration

In this section, we produce examples of Galois covers which do not have a good degeneration. Let f be an odd prime and put $\alpha := 2$. Define $m := \alpha^f - 1 = 1 + \alpha + \cdots + \alpha^{f-1}$. We define

$$\mathbf{a} = (1, \alpha, \alpha^2, \ldots, \alpha^{f-1}).$$

We suppose that $(X = \mathbb{P}^1_k; \mathbf{x})$ is the generic f-pointed curve of genus zero and let $g : Z \to X = \mathbb{P}^1_k$ be the m-cyclic cover of type $(\mathbf{x}; \mathbf{a})$.

As in §1, we define $\gamma(s) = (\sum_{t=0}^{f-1} \langle s\alpha^t \rangle)/m$.

Lemma 3.1 *Let $S \subsetneq \{0, \ldots, f-1\}$ and $s := \sum_{j\in S} \alpha^j$. Then*

$$\gamma(s) = |S|, \qquad \gamma(m-s) = f - |S|.$$

Proof: Let s be as in the statement of the lemma. The definition of m implies that $\alpha^f \equiv 1 \pmod m$. Therefore

$$\langle s\alpha^i \rangle = \sum_{j\in S} \alpha^{i+j},$$

where the powers of α should be read modulo f. This implies that

$$\gamma(s) = \frac{1}{m} \sum_{t=0}^{f-1} \sum_{j\in S} \langle \alpha^{j+t} \rangle = \frac{1}{m} \sum_{j\in S} (1 + \alpha + \cdots + \alpha^{f-1}) = |S|.$$

The second statement follows immediately from the first statement and the definition of m. □

Lemma 3.2 *Let $p \geq m(f-3)$ be a prime such that $p^f \equiv 1 \pmod m$. Then Z is ordinary, i.e.*

$$b := B(\mathbf{a}) = g(Z) = (f-1)(m-1)/2.$$

In particular, there exists a tame $G := (\mathbb{Z}/p\mathbb{Z})^b \rtimes \mathbb{Z}/m\mathbb{Z}$-Galois cover $\pi : Y \to \mathbb{P}^1_k$ of type $\mathbf{a}$.

Proof: The assumption on p implies that $p \equiv \alpha^i \pmod m$, for some i. Therefore $\gamma(sp^i) = \gamma(s)$, for all i. The statement now follows immediately from Corollary 1.2. □

We now suppose that the order of p in $\mathbb{Z}/m\mathbb{Z}^*$ is f. The goal of this section is to show that the cover π from Lemma 3.2 does not have a good degeneration. It suffices to show that every degeneration $g_0 : Z_0 \to X_0$ of $g : Z \to X$ is nonordinary. As remarked in §2, it suffices to consider degenerations $g_0 : Z_0 \to X_0$ of $g : Z \to X$ such that fiber X_0 consists of two irreducible components X_1 and X_2 intersecting in one point τ.

Consider such a degeneration $g_0 : Z_0 \to X_0$. We let $S_i \subset \{1, \ldots, f\}$ be the subset of indices j such that x_j specializes to X_i $(i = 1, 2)$. We may assume that $2 \leq |S_i| \leq f-2$.

Proposition 3.3 *Let $g_0 : Z_0 \to X_0$ be a degeneration of $g : Z \to X$. Then Z_0 is nonordinary.*

Proof: We assume that X_0 consists of two irreducible components X_1 and X_2 which intersect in one point τ, and let S_i be as above. We write $g_i : Z_i \to X_i$ for the restriction of g_0 to X_i. The curve Z_0 is ordinary if and only if Z_i is ordinary, for $i = 1, 2$.

We define

$$\gamma_1(i) = \frac{1}{m}\left(\langle \sum_{j \notin S_1} i\alpha^j \rangle + \sum_{j \in S_1} \langle \alpha^j \rangle).\right)$$

These are the terms occuring in the bound for the cover $g_1 : Z_1 \to X_1$ (Lemma 2.2). Note that g_1 is branched at $r_1 + 1$ points, namely the specialization of x_j for $j \in S_1$ and the singular point τ. It follows from Corollary 1.2 that Z_1 is ordinary if and only if $\gamma_1(i) = \gamma_1(p^j i)$, for all j.

Let $s = \sum_{j \in S_1} \alpha^j$. Put

$$d_s = \gamma_1(s) - \sum_{j \in S_1} \langle s\alpha^j \rangle = \langle s \sum_{j \notin S_1} \alpha^j \rangle = \langle s(m - s) \rangle = \langle m - s^2 \rangle.$$

Then Lemma 3.1 implies that

$$\gamma(d_s) = \gamma(m - s^2) = f - \gamma(s^2).$$

Since f is odd, it follows that $\gamma(s^2) = \gamma(s)$. We conclude that

$$\gamma(d_s) = f - \gamma(s) = f - |S_1|.$$

We claim that there exists an i such that $\gamma_1(sp^i) \neq \gamma_1(s)$. Namely,

$$\sum_{i=0}^{f-1} \gamma_1(s\alpha^i) = |S_1|^2 + f - |S_1| \equiv |S_1|(|S_1| - 1) \pmod{f}.$$

Since $2 \leq |S_1| \leq f - 2$, we conclude therefore that $\sum_{i=0}^{f-1} \gamma_1(s\alpha^i) \not\equiv 0 \pmod{f}$. This shows that there exists an i such that $\gamma_1(p^i s) \neq \gamma_1(s)$. Hence Z_1 is not ordinary. □

As already remarked above, the following corollary immediately follows from Proposition 3.3.

Corollary 3.4 *Let $\pi : Y \to \mathbb{P}^1$ be as in Corollary 1.2. Then π does not have a good degeneration.*

Remark 3.5 It seems that Proposition 3.3 is a special case of a much more general statement. Let m be an odd integer and let α be an element of order $n|(m-1)/2$ in $\mathbb{Z}/m\mathbb{Z}^*$. Let p be sufficiently large such that p has order $f|n$ in $\mathbb{Z}/m\mathbb{Z}^*$. Let $g : Z \to X$ be an m-cyclic cover of type $(\mathbf{x}; \mathbf{a})$, where $(X; \mathbf{x})$ is the generic n-pointed curve of genus 0 and $\mathbf{a} = (1, \alpha, \ldots, \alpha^{n-1})$. As in the proof of Corollary 1.2, one checks that Z is ordinary. Explicit computations suggest that the tame $(\mathbb{Z}/p\mathbb{Z})^{g(Z)} \rtimes \mathbb{Z}/m\mathbb{Z}$-Galois cover corresponding to g does not have a good degeneration if f is odd and strictly larger than 1. This suggests that for $f \geq 3$ there is no generalization of the statement of Remark 2.4.

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